Method of initial functions for beams

Patel Rakesh, Dubey S.K., Pathak K.K.

Abstract— In this paper, method of initial functions (MIF) is used for the analysis of concrete beams. The equations of two dimensional elasticity have been used for deriving the governing equations. Numerical solutions of the governing equations have been presented for simply supported beam loaded with uniformly distributed load. Two cases with varying depth to span ratio have been considered for analysis. The method of initial function is an analytical method of elasticity theory. No assumptions regarding the distribution of stress or displacements are needed in this method. This method can be applied for beam of any depth and loading.

Index Terms—Beams, method of initial functions, stress, strain, displacement, Isotropic, Elasticity,

1 INTRODUCTION

The beam theories which are based on certain assumptions regarding the distribution of stresses and displacements are of a practical utility in the case of those problems, where the beam thickness is less. The results obtained by these theories are away from actual physical behavior of flexural members. The two major theories generally used for the beam analysis, The Bernoulli-Euler theory of bending and Timoshenko beam theory are based on assumptions. One common assumption is that transverse sections which are plane before bending remain plane after bending. However, it has been observed that beam sections especially in the case of deep beams, warp under loaded conditions. So in the problems involving thick beams and layered beams it becomes difficult to obtain useful results using these theories.

The method of initial function is an analytical method of elasticity theory. The method makes it possible to obtain exact solutions of different types of problems, i.e., solutions without the use of hypotheses about the character of stress and strain. According to this method, the basic desired functions are the displacements and stresses, the system of differential equations which are obtained from equations of Hook's law and equilibrium equations by replacing stresses by the displacements according to elasticity relations. The order of the derived equations depends on the stage at which the series representing the stresses and displacements are truncated.

From the literature we have observed that this method has various applications in structural engineering but very few researchers have used MIF for beams. A method for solving problems of theory of elasticity for the analysis thick plates as well as shells which is known as the method of initial functions [1]. The method of initial function has been applied for deriving higher order theories for laminated composite thick rectangular plates [2]. It is used for the analysis of rectangular and long beams [3],[4]. MIF is applied for the analysis of orthotropic deep beams [5].

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There so many other theories which we are using for the analysis of beams.Developed Hyperbolic Shear Deformation Theory for transverse shear deformation effects. It is used for the static flexure analysis of thick isotropic beams [6]. A layer wise trigonometric shear deformation theory is used for the analysis of two layered cross ply laminated simply supported and fixed beams subjected to sinusoidal load [7].

The displacements and stresses of the beam can be represented by the angle of rotation and the deflection of the neutral surface. Based on the refined beam theory, the exact equations for the beam without transverse surface loadings are derived [8].

2 FORMULATION OF MIF

According to this method, the basic desired functions are the displacements and stresses, the system of differential equations which are obtained from equations of Hook's law and equilibrium equations by replacing stresses by the displacements according to elasticity relations. The order of the derived equations depends on the stage at which the series representing the stresses and displacements are truncated.

The equations of equilibrium for solids ignoring the body forces for two-dimensional case are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad (1)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \quad (2)$$

The stress-strain relations for isotropic material are:

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$$\sigma_x = C_{11}\varepsilon_x + C_{12}\varepsilon_y \quad (3)$$

$$\sigma_y = C_{12}\varepsilon_x + C_{22}\varepsilon_y \quad (4)$$

$$\tau_{xy} = C_{33}\gamma_{xy} \quad (5)$$

The values of the coefficients C_{11} to C_{33} for isotropic materials are given in Appendix.

The strain displacement relations for small displacements are:

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad (6)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \quad (7)$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad (8)$$

Eliminating σ_x between equations (1) and (2) the following equations are obtained, which can be written in matrix form as;

$$\frac{\partial}{\partial y}\begin{bmatrix} u \\ v \\ Y \\ X \end{bmatrix} = \begin{bmatrix} 0 & -\alpha & 0 & 1/G \\ C_1\alpha & 0 & C_2 & 0 \\ 0 & 0 & 0 & -\alpha \\ C_3\alpha^2 G & 0 & C_1\alpha & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ Y \\ X \end{bmatrix} \quad (9)$$

Where,

$$X = \tau_{xy}, \quad Y = \sigma_y = C_{12}\varepsilon_x + C_{22}\varepsilon_y$$

$$C_1 = \frac{-a_{12}}{a_{22}}; C_2 = \frac{1}{Ga_{22}}; C_3 = \frac{a_{12}}{a_{22}} - a_{11}$$

and

$$a_{11} = \frac{C_{11}}{G}, a_{12} = \frac{C_{12}}{G}, a_{22} = \frac{C_{22}}{G}$$

The equation (9) can be expressed as:

$$\frac{\partial}{\partial y}\{S\} = [D]\{S\} \quad (10)$$

The solution of equation (10) is

$$\{S\} = \left[e^{[D]y}\right]\{S_0\} \quad (11)$$

Where $\{S_0\}$ is the vector of initial functions, being the value of the state vector $\{S\}$ on the initial plane.
If u_0, v_0, Y_0 and X_0 are values of u, v, Y and X respectively, on the initial plane, then

$$\{S_0\} = [u_0, \quad v_0, \quad Y_0, \quad X_0]^T \quad (12)$$

$$[L] = e^{[D]y} \quad (13)$$

Expending (13) in the form of a series

$$[L] = [I] + y[D] + \frac{y^2}{2!}[D]^2 + ... \quad (14)$$

3 APPLICATION OF MIF

An isotropic beam of length l, depth, H and loaded with uniformly distributed load p in the y- direction. The bottom plane of the beam is taken as the initial plane. Due to loading at the top plane of the beam one has
$X_0 = Y_0 = 0$

On the plane, y = H, the conditions are
X = 0, Y = -p
Y= -p on y = H,

After simplification yields the governing partial differential equation:

$$(L_{Yu}.L_{Xv} - L_{Yv}.L_{Xu})\phi = -p \quad (15)$$

Initial functions are obtained by substituting the value of Φ:

$$u_0 = L_{Xv}\phi \quad (16)$$

$$v_0 = -L_{Xu}\phi \quad (17)$$

From the value of initial functions the value of displacements and stresses are obtained.

$$\begin{aligned} u &= L_{uu}.u_0 + L_{uv}.v_0 \\ v &= L_{vu}.u_0 + L_{vv}.v_0 \\ Y &= L_{Yu}.u_0 + L_{Yv}.v_0 \\ X &= L_{Xu}.u_0 + L_{Xv}.v_0 \end{aligned} \quad (18)$$

4 NUMERICAL EXAMPLE

The following values of beam dimensions are chosen for the particular problem,
H =1000 mm and 2000 mm , l = 4000 mm
The following material properties are taken:
E=2.10x105 N/mm², μ = 0.30, G = 0.10x105 N/mm²
The boundary conditions of the simply supported edges are:
X = Y = v = 0, at x = 0 and x = l

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The boundary conditions are exactly satisfied by the auxiliary function.

$\Phi = A_1 \sin(\pi x/l)$

A uniformly distributed load p = 25N/mm is assumed, on the top surface of the beam.

5 RESULTS AND DISCUSSION

The value of auxiliary function Φ is obtained from equation (15) using this value of auxiliary function, the values of initial functions u_0 and v_0 is obtained from equation (16) and (17).

The initial functions are operated upon by the transfer matrix successively across each layer until the entire beam is analysed and the stresses at the top surface are again obtained. Governing equation (15) of desired order according to the requirements of a beam problem is obtained using MIF.

The values of u_0 and v_0 are substituted in expression (18) for obtaining the values of stresses and displacements. The distribution of stresses and displacements across the depth of a simply supported beam for uniformly distributed load are shown in figure 1 to 10.

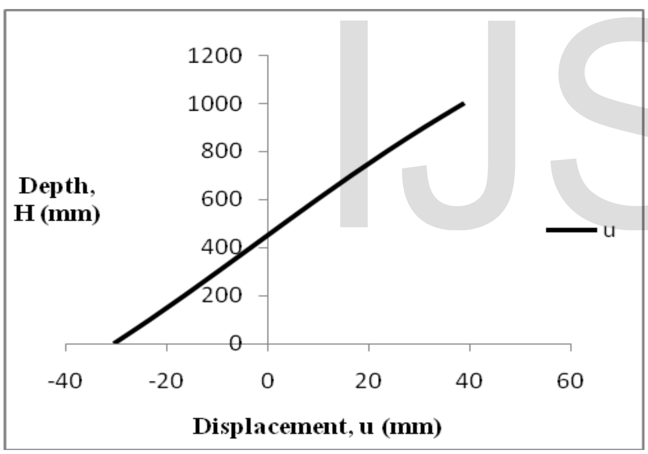

Figure 1. Variation of "u" through the thickness of beam for H/l= 0.25

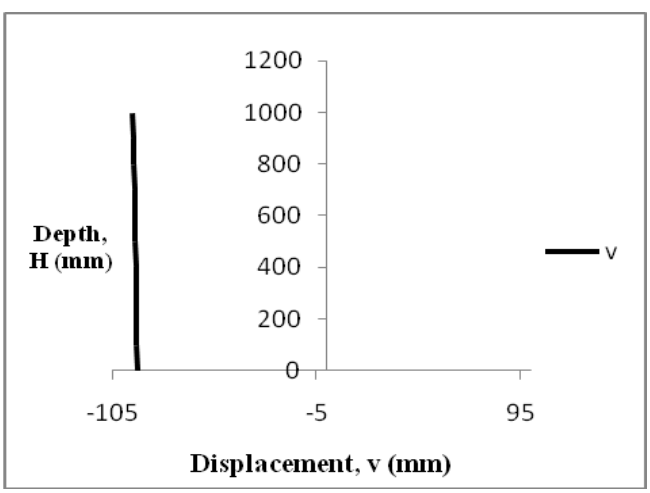

Figure 2. Variation of "v" through the thickness of beam for H/l= 0.25

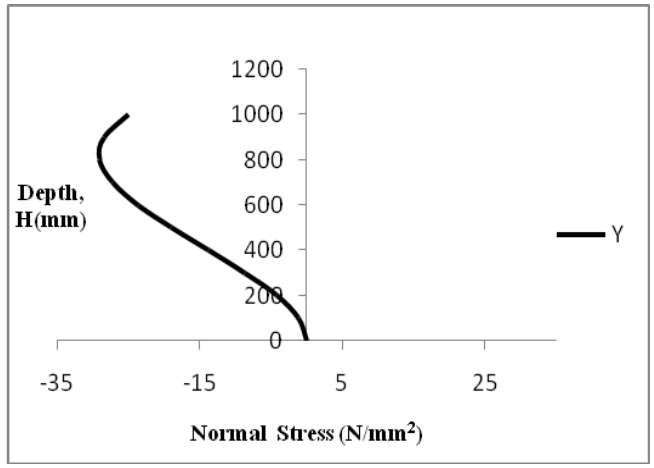

Figure 3. Variation of "Normal stress (Y)" through the thickness of beam for H/l= 0.25

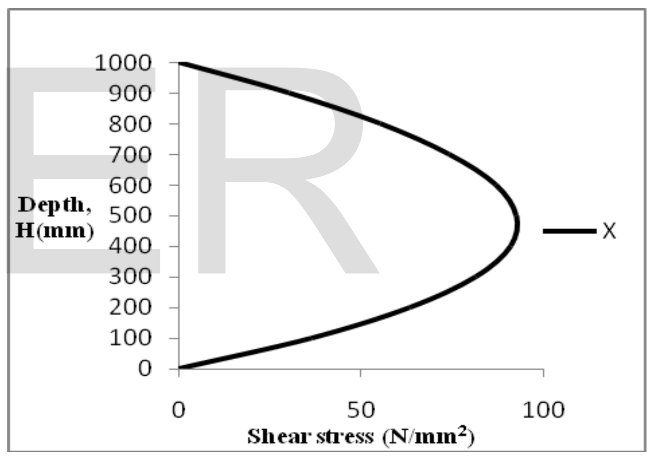

Figure 4. Variation of "Shear stress (X)" through the thickness of beam for H/l= 0.25

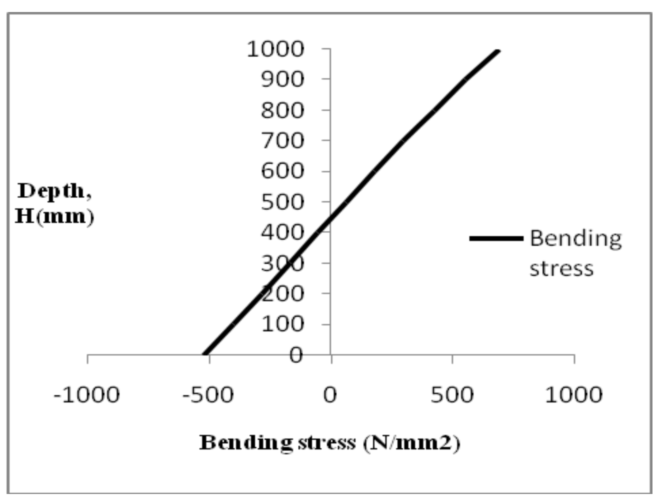

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Figure 5. Variation of "Bending tress (σ_x)" through the thickness of beam for H/l= 0.25

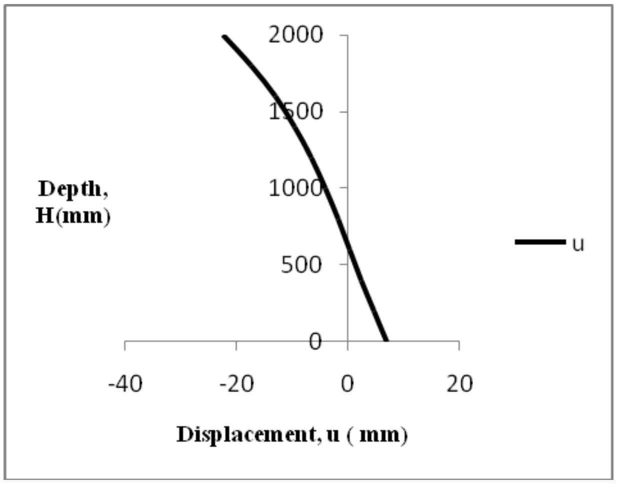

Figure 6 Variation of "u" through the thickness of beam for H/l= 0.50

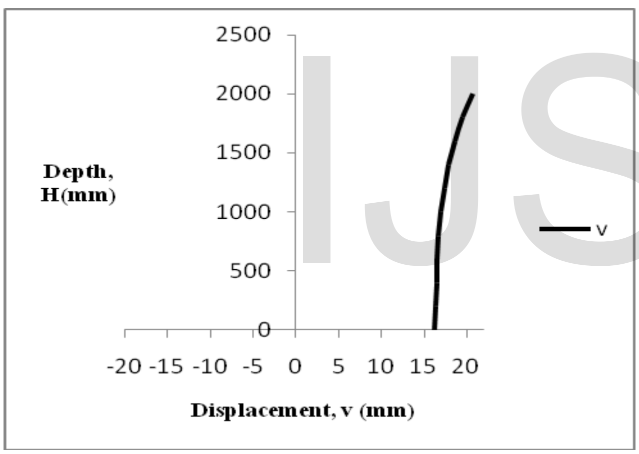

Figure 7. Variation of "v" through the thickness of beam for H/l= 0.50

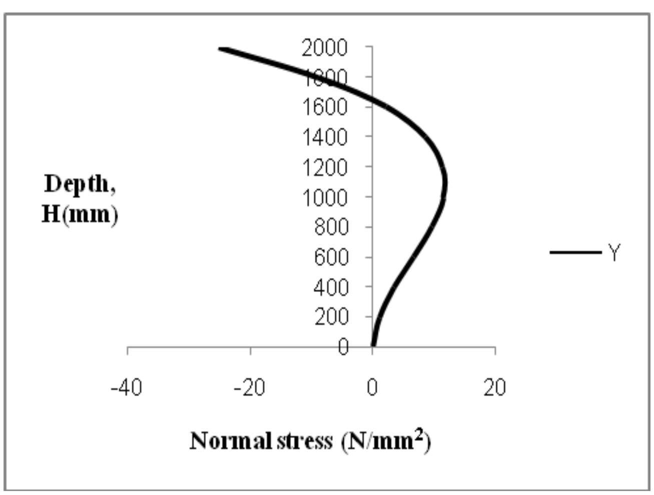

Figure 8. Variation of "Normal stress (Y)" through the thickness of beam for H/l= 0.50

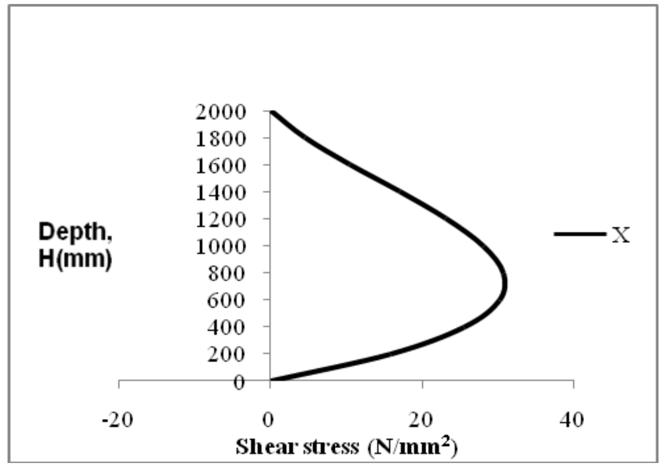

Figure 9. Variation of "Shear stress (X)" through the thickness of beam for H/l= 0.50

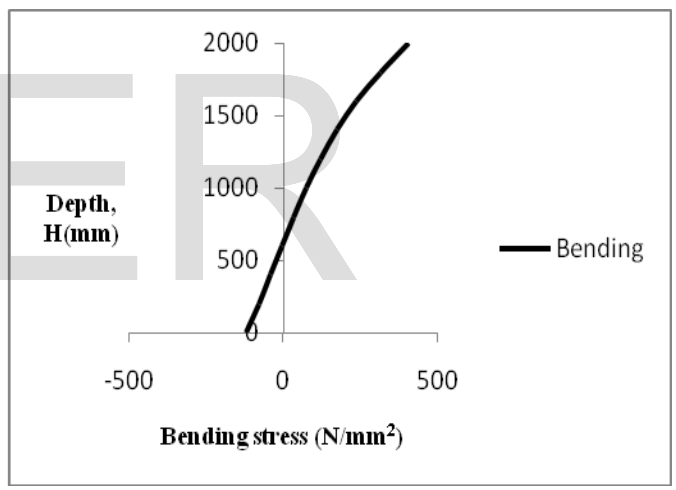

Figure 10. Variation of "Bending tress (σ_x)" through the thickness of beam for H/l= 0.50

From figures 1 and 5 that the value of displacement 'u' is more at the top surface and less at the bottom surface. For H/l= 0.25 the variation of 'u' is almost linear but for H/l=0.50 its variation is more at the top surface of beam.
It can be seen from figures 2 and 7 that displacement 'v' is uniform throughout the depth for H/l= 0.25 and there is small variation for H/l=0.50.

From figures 3 and 8 it is observed that the value of normal stress (Y) is zero at the bottom and maximum at the top of beam. The physical condition of normal stress equal to the applied normal load at the top fibre is satisfied.

It is seen from figures 4 and 9 that the shear stress (X) is maximum at mid depth in case of H/l= 0.25 and it is just below the mid depth in case of H/l=0.50.

From figures 5 and 10 it is observed that the variation of bending stress is almost linear in case of H/l= 0.25 but in the case of H/l=0.50 it is not linear near the top surface of the beam. The neutral axis shifts from mid depth towards bottom surface. It shows that the deep beam action comes into effect when H/l= 0.25.

6 CONCLUSIONS

The nature of the curves obtained for stresses and displacements is similar to those obtained by other theories. Deep beam action is clearly seen at H/= 0.25 and 0.50. MIF yields correct results for both shallow and deep beam. In the theories based on assumptions this effect is not seen. No correction factor is required in the case of deep beam.Hence it can be successfully used as an alternative approach for the analysis of beams. It also gives accurate results in case of small thickness, large thickness and layered members. In this method no assumption regarding the position of neutral axis is required.

REFERENCES

[1] Vlasov V Z., "Method of initial functions in problems of theory of thick plates and shells", Proceedings of 9th International conference of Applied Mechanics, University of Brussels, Belgium, 6,1957, pp 321-330.

[2] Iyengar, K.T.S. and Pandya, S.K. (1986). Application of the method of initial functions for the analysis of composite laminated plates. Archive of Applied Mechanics.56 (6): 407-416.

[3] Iyengar, KTS, Chandrashekhara K and Sebastian V.K., "Thick Rectangular Beams", Journal of the Engineering Mechanics Division-ASCE, Vol.100 No.6,1974, pp 1277-1282.

[4] Iyengar, K., Raja, T. S, and Raman, P. V. (1979).Free vibration of rectangular beams of arbitrary depth, Acta Mechanica 32 (1): 249-259.

[5] Dubey, S.K. (2005). Analysis of homogeneous orthotropic deep beams. Journal of Structural Engineering 32(2): 109-166.

[6] Ghugal, Y.M.and Sharma Rajneesh (2011). A refined shear deformation theory for flexure of thick beams. Latin American Journal of Solids and Structures 8:183-195.

[7] Ghugal, Y.M.and Shinde, S.B. (2013). Flexural analysis of cross-ply laminated beams using layer wise trigonometric shear deformation theory. Latin American Journal of Solids and Structures 10: 675 – 705.

[8] Yang GAO & Minzhong WANG. (2006).The refined theory of deep rectangular beams based on general solutions of elasticity. Science in China: Series G Physics, Mechanics & Astronomy. 49(3): 291-303.

Notation

l - Effective span of beam
d - Total thickness of beam
E - Young's modulus of Elasticity
G - Shear modulus of Elasticity
μ - Poisson's ratio
ε - Strain
σ_x - Bending stress
σ_y - Normal stress
τ_{xy} - Shear stress
u - Displacements in x directions
v- Displacements in y directions
$\alpha - \frac{\partial}{\partial x}$